Mellin transforms and asymptotics: Harmonic sums

Phillipe Flajolet,
Xavier Gourdon,
Philippe Dumas

Die Theorie der reziproken Funktionen und Integrale ist ein centrales Gebiet, welches manche anderen Gebiete der Analysis miteinander verbindet.

Hjalmar Mellin

Reporter :
Ilya Posov
Saint-Petersburg State University
Faculty of Mathematics and Mechanics

Joint Advanced Student School 2004

Contents

- Mellin transform and its basic properties
- Direct mapping
- Converse mapping
- Harmonic sums

Robert Hjalmar Mellin

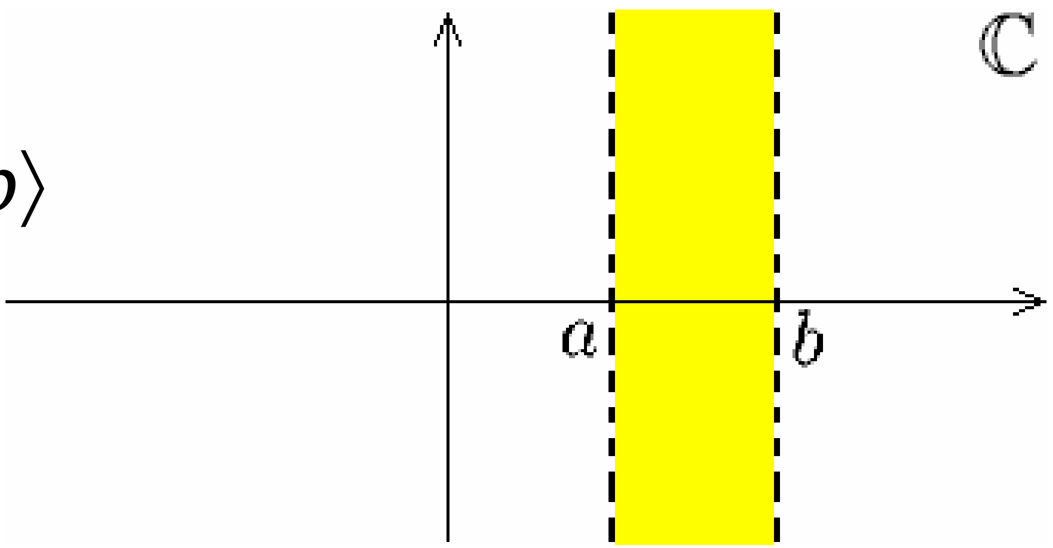
Born: 1854 in Liminka, Northern Ostrobothnia, Finland
Died: 1933

Some terminology

- Analytic function $f(s) = c_0 + c_1(s - s_0) + c_2(s - s_0)^2 + \cdots$

- Meromorphic function

- Open strip $\langle a, b \rangle$

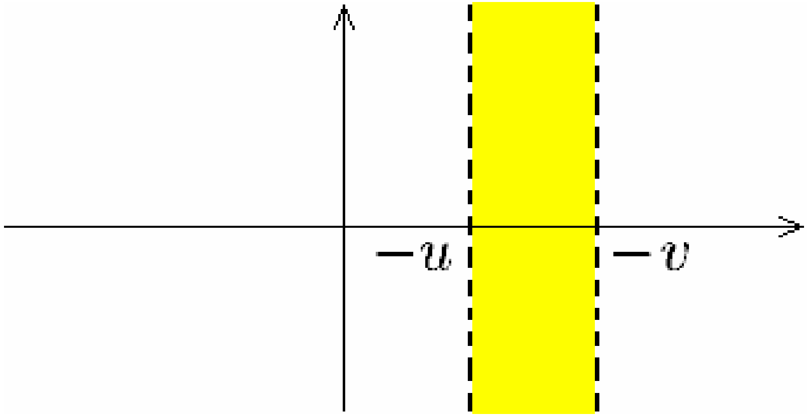


Mellin transform

$$M[f(x);s] = f^*(s) = \int_0^\infty f(x)x^{s-1}dx$$

$$f(x) = O(x^u) \quad x \to 0$$

$$f(x) = O(x^v) \quad x \to +\infty$$



fundamental strip

$f^*(s)$ exists in the strip $\langle -u, -v \rangle$

Mellin transform example

$$f(x) = \frac{1}{1+x}$$

$$\frac{1}{1+x} = \mathrm{O}(1) \quad x \to 0$$

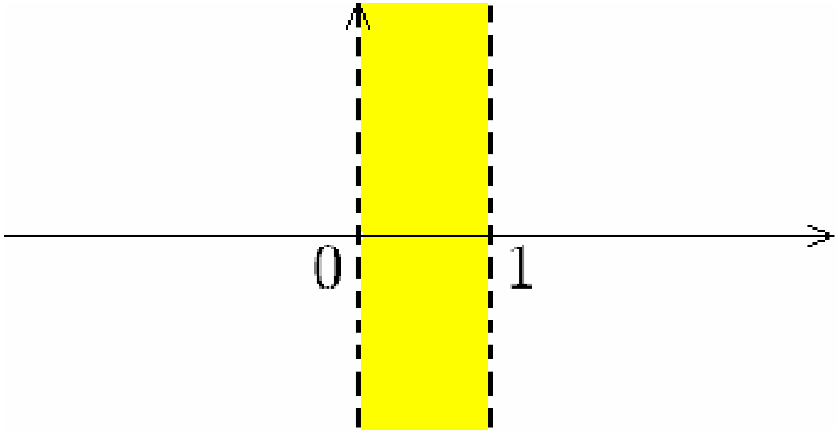
$$\frac{1}{1+x} = \mathrm{O}(x^{-1}) \quad x \to +\infty$$

$$f^{*}(s) = \int_0^{+\infty} \frac{x^{s-1}}{1+x} dx = \frac{\pi}{\sin \pi s}$$

fundamental strip :

$$u = 0 \quad v = -1$$

$$\langle -u, -v \rangle = \langle 0, 1 \rangle$$



Gamma function

$$f(x) = e^{-x}$$

$$e^{-x} = O(1) \quad x \to 0$$

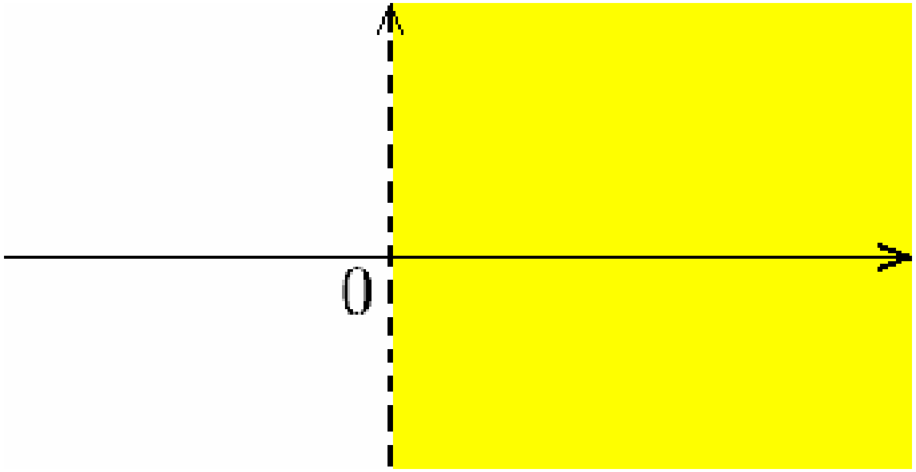
$$e^{-x} = O(x^{-b}) \; \forall b > 0 \quad x \to +\infty$$

$$f^*(s) = \int_0^{+\infty} e^{-x} x^{s-1} dx = \Gamma(s) \qquad s\Gamma(s) = \Gamma(s+1)$$

fundamental strip :

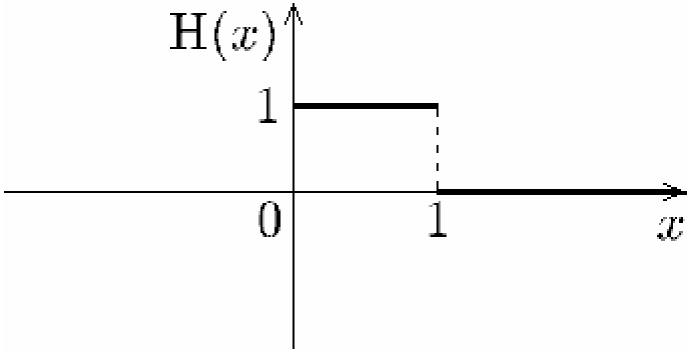
$$u = 0 \quad v = -\infty$$

$$\langle -u, -v \rangle = \langle 0, +\infty \rangle$$



Transform of step function

$$\mathrm{H}(x)=\begin{cases}0, & x\in[0,1]\\ 1, & x\in(1,+\infty)\end{cases}$$



$$\mathrm{H}(x)=\mathrm{O}(1)\quad x\to 0$$

$$\mathrm{H}(x)=\mathrm{O}(x^{-b})\ \forall b>0\quad x\to+\infty$$

$$\mathrm{H}^*(x)=\int_0^{+\infty}H(x)x^{s-1}dx=\int_0^1 x^{s-1}dx=\frac{1}{s},\quad s\in\langle 0,+\infty\rangle$$

$$\overline{\mathrm{H}}(x)=1-\mathrm{H}(x),\quad \overline{\mathrm{H}}^*(x)=-\frac{1}{s},\quad s\in\langle-\infty,0\rangle$$

Basic properties (1/2)

$f(x)$	$f^*(s)$	$\langle \alpha, \beta \rangle$	
$x^\nu f(x)$	$f^*(s+\nu)$	$\langle \alpha-\nu, \beta-\nu \rangle$	
$f(x^\rho)$	$\frac{1}{\rho} f^*\left(\frac{s}{\rho}\right)$	$\langle \rho\alpha, \rho\beta \rangle$	$\rho > 0$
$f(1/x)$	$-f^*(-s)$	$\langle -\beta, -\alpha \rangle$	
$f(\mu x)$	$\frac{1}{\mu^s} f^*(s)$	$\langle \alpha, \beta \rangle$	$\mu > 0$
$\sum_k \lambda_k f(\mu_k x)$	$\left(\sum_k \lambda_k \mu^{-s}\right) f^*(s)$		

Basic properties (2/2)

$f(x)$	$f^*(s)$	$\langle \alpha, \beta \rangle$	
$f(x)\log x$	$\frac{d}{ds} f^*(s)$	$\langle \alpha, \beta \rangle$	
$\Theta f(x)$	$-sf^*(s)$	$\langle \alpha', \beta' \rangle$	$\Theta = x\frac{d}{dx}$
$\frac{d}{dx} f(x)$	$-(s-1) f^*(s-1)$	$\langle \alpha'+1, \beta'+1 \rangle$	
$\int_0^x f(t)dt$	$-\frac{1}{s} f^*(s+1)$		

Zeta function

$$M\left[\sum_k \lambda_k f(\mu_k x); s\right] = \left(\sum_k \lambda_k \mu^{-s}\right) f^*(s)$$

$$g(x) = \frac{e^{-x}}{1-e^{-x}} = e^{-x} + e^{-2x} + e^{-3x} + \ldots$$

$$\lambda_k = 1, \quad \mu_k = k, \quad f(x) = e^{-x}$$

$$g^*(s) = \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) M\left[e^{-x}; s\right] = \zeta(s)\Gamma(s)$$

$$s \in \langle 1, +\infty \rangle$$

Some Mellin tranforms

e^{-x}	$\Gamma(s)$	$\langle 0, +\infty \rangle$
$e^{-x} - 1$	$\Gamma(s)$	$\langle -1, 0 \rangle$
e^{-x^2}	$\frac{1}{2}\Gamma\left(\frac{1}{2}s\right)$	$\langle 0, +\infty \rangle$
$\dfrac{e^{-x}}{1-e^{-x}}$	$\zeta(s)\Gamma(s)$	$\langle 1, +\infty \rangle$
$\dfrac{1}{1+x}$	$\dfrac{\pi}{\sin \pi s}$	$\langle 0, 1 \rangle$
$\log(1+x)$	$\dfrac{\pi}{s \sin \pi s}$	$\langle -1, 0 \rangle$
$\mathrm{H}(x) \equiv 1_{0<x<1}$	$\dfrac{1}{s}$	$\langle 0, +\infty \rangle$
$x^{\alpha} (\log x)^k \, \mathrm{H}(x)$	$\dfrac{(-1)^k k!}{(s+\alpha)^{k+1}}$	$\langle -\alpha, +\infty \rangle \; k \in \mathbb{N}$

Inversion

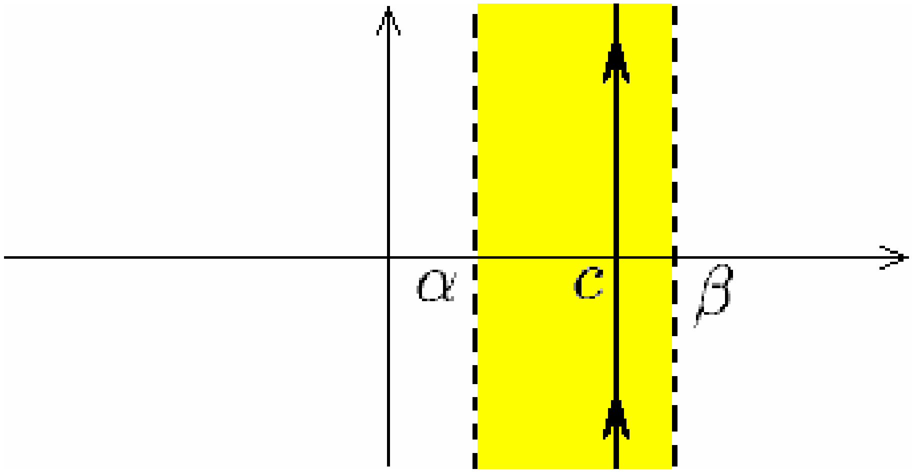
Theorem 1

Let $f(x)$ be integrable with fundamential strip $\langle \alpha, \beta \rangle$.

Let $\alpha < c < \beta$ and $f^*(c + \mathrm{i}t)$ is integrable, then the equality

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} f^*(s) x^{-s} ds = f(x)$$

holds almost everywhere.



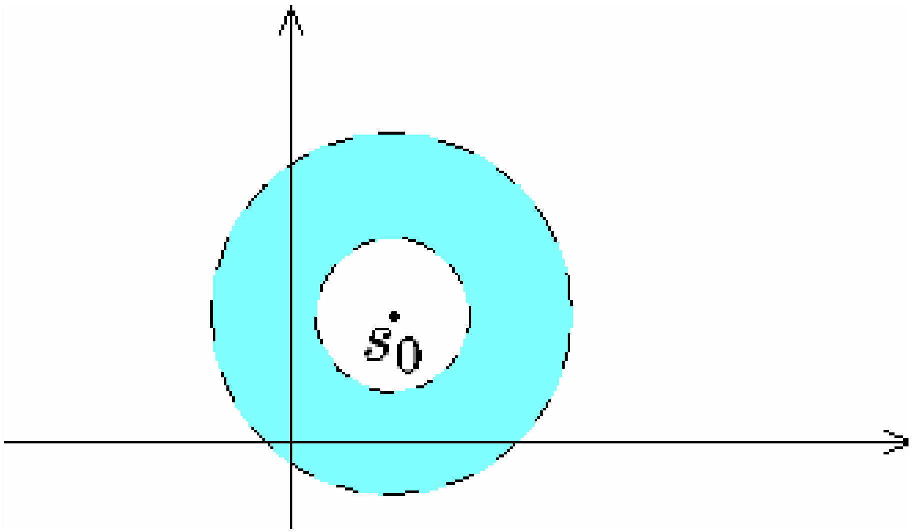
Laurent expansion

$$\phi(s) = \sum_{k \geq -r}^{+\infty} c_k (s - s_0)^k$$

$$c_{-r} \neq 0$$

Pole of order r if r>0

Analytic in s_0 if $r \leq 0$



Example :

$$\frac{1}{s^2(s+1)} = \frac{1}{s+1} + 2 + 3(s+1) + \cdots \quad (s_0 = -1)$$

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + 1 + \cdots \quad (s_0 = 0)$$

Singular element

Definition 1

A singular element of $\phi(s)$ at s_0 is an initial sum of Laurent expansion truncated at terms of $\mathrm{O}(1)$ or smaller :

$$\phi(s) = \frac{1}{s^2(s+1)} \underset{s\to 0}{=} \frac{1}{s^2} - \frac{1}{s} + 1 - \cdots \underset{s\to -1}{=} \frac{1}{s+1} + 2 + 3(s+1) + \cdots$$

s.e. at $s_0 = 0$: $\left[\frac{1}{s^2} - \frac{1}{s}\right], \left[\frac{1}{s^2} - \frac{1}{s} + 1\right], \ldots$

s.e. at $s_0 = -1$: $\left[\frac{1}{s+1}\right], \left[\frac{1}{s+1} + 2\right], \left[\frac{1}{s+1} + 2 + 3(s+1)\right], \ldots$

Singular expansion

Definition 2

Let $\phi(s)$ be meromorfic in Ω with $\wp$ including all the poles of $\phi(s)$ in Ω. A singular expansion of $\phi(s)$ in Ω is a formal sum of singular elements of $\phi(s)$ at all points of $\wp$

Notation : $\phi(s) \asymp E \quad (s \in \Omega)$

$$\frac{1}{s^2(s+1)} \asymp \left[\frac{1}{s+1}\right]_{s=-1} + \left[\frac{1}{s^2} - \frac{1}{s}\right]_{s=0} + \left[\frac{1}{2}\right]_{s=1} \qquad s \in \langle -2, 2 \rangle$$

Singular expansion of gamma function

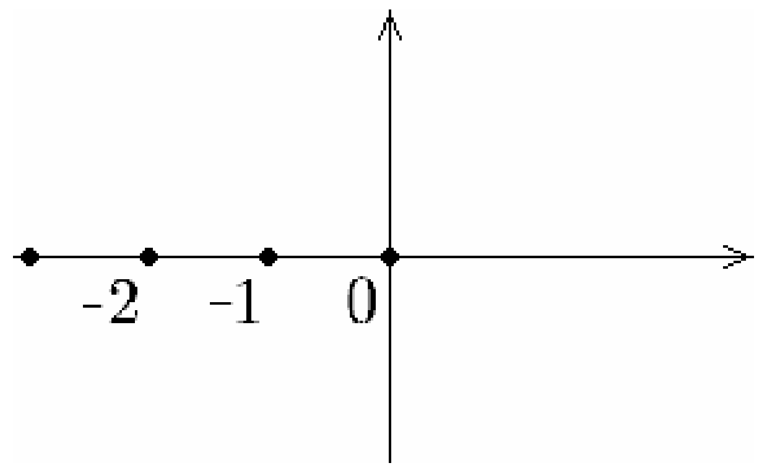
$$s\Gamma(s) = \Gamma(s+1)$$

$$\Gamma(s) = \frac{\Gamma(s+m+1)}{s(s+1)(s+2)\cdots(s+m)}$$

Thus $\Gamma(s)$ has poles at the points $s = -m$ with $m \in \square$

$$\Gamma(s) \square \frac{(-1)^m}{m!}\frac{1}{s+m}$$

$$\Gamma(s) \asymp \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!}\frac{1}{s+k} \quad s \in \square$$



poles of gamma function

Direct mapping (1/3)

Theorem 2

Let $f(x)$ have a transform $f^*(s)$ with nonempty fundamental strip $\langle \alpha, \beta \rangle$.

$$f(x) = \sum_{(\xi,k)\in A} c_{\xi,k} x^{\xi} (\log x)^k + \mathrm{O}(x^{\gamma}) \quad x \to 0^+$$

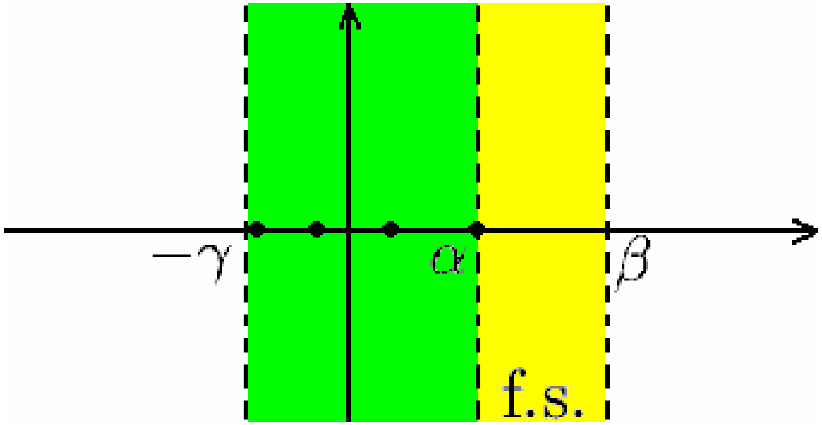
$$k \in \mathbb{N}, \quad -\gamma < -\xi \leq \alpha$$

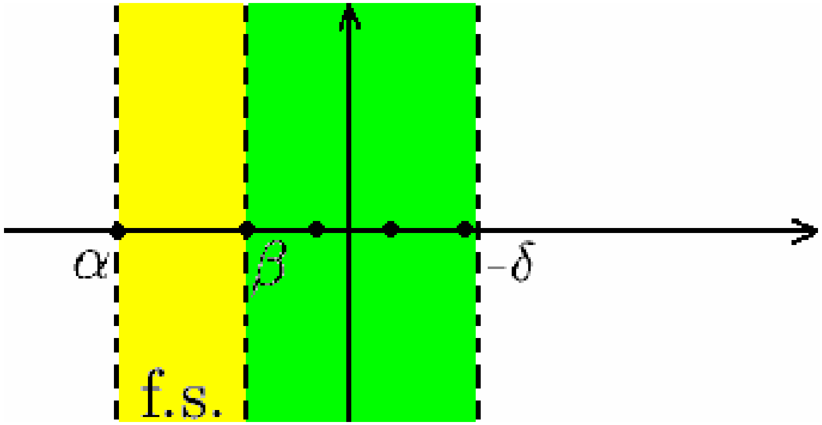
Then $f^*(s)$ is continuable to a meromorphic function in the strip $\langle -\gamma, \beta \rangle$ where it admits the singular expansion

$$f^*(s) \asymp \sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} \quad (s \in \langle -\gamma, \beta \rangle)$$

Direct mapping (2/3)

$f(x)$	$f^*(s)$
Order at 0: $O(x^{-\alpha})$	Leftmost boundary of f.s. at $\Re(s) = \alpha$
Order at $+\infty$: $O(x^{-\beta})$	Rightmost boundary of f.s. at $\Re(s) = \beta$
Expansion till $O(x^{\gamma})$ at 0	Meromorphic continuation till $\Re(s) \geq -\gamma$
Expansion till $O(x^{\delta})$ at $+\infty$	Meromorphic continuation till $\Re(s) \leq -\delta$





Direct mapping (3/3)

$f(x)$	$f^*(s)$
Term $x^a(\log x)^k$ at 0	Pole with s.e. $\dfrac{(-1)^k k!}{(s+a)^{k+1}}$
Term $x^a(\log x)^k$ at $+\infty$	Pole with s.e. $-\dfrac{(-1)^k k!}{(s+a)^{k+1}}$
Term x^a at 0	Pole with s.e. $\dfrac{1}{s+a}$
Term $x^a \log x$ at 0	Pole with s.e. $-\dfrac{1}{(s+a)^2}$

Example 0

$$f(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \underset{x \to 0}{=} \sum_{j=0}^{M} \frac{(-1)^j}{j!} x^j + \mathrm{O}\left(x^{M+1}\right)$$

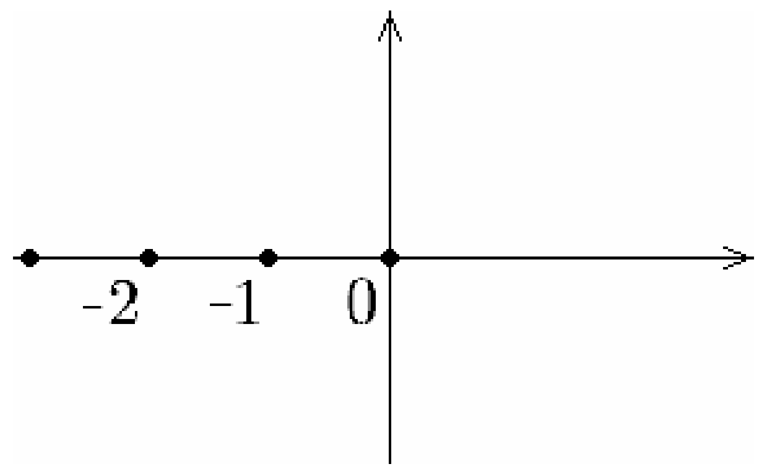
$$f^*(s) = \Gamma(s), \quad s \in \langle 0, +\infty \rangle$$

$f^*(s)$ is meromorphically continuable to $\langle -M-1, +\infty \rangle$

$$f^*(s) \asymp \sum_{j=0}^{\mathrm{M}} \frac{(-1)^j}{j!} \frac{1}{s+j} \quad s \in \langle -M-1, +\infty \rangle$$

finally

$$\Gamma(s) \asymp \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \frac{1}{s+j} \quad s \in \mathbb{C}$$



poles of gamma function

Example 1

$$f(x) = \frac{e^{-x}}{1-e^{-x}}, \quad f^*(s) = \Gamma(s)\zeta(s) \quad s \in \langle 1, +\infty \rangle$$

$$\frac{e^{-x}}{1-e^{-x}} \underset{x \to 0}{=} \sum_{j=-1}^{+\infty} B_{j+1} \frac{x^j}{(j+1)!}, \quad B_0 = 1, B_1 = -\frac{1}{2}, \cdots$$

$$\Gamma(s)\zeta(s) \asymp \sum_{j=-1}^{+\infty} \frac{B_{j+1}}{(j+1)!} \frac{1}{s+j} \quad s \in \square$$

$$\Gamma(s) \asymp \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \frac{1}{s+j} \quad s \in \square$$

$$\zeta(s) \asymp \frac{1}{s-1} \quad s \in \square, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta(-m) = -\frac{B_{m+1}}{m+1}$$

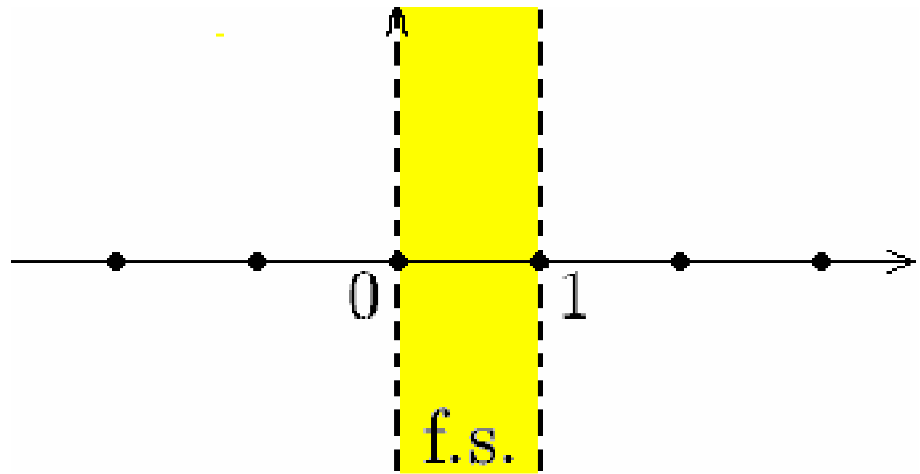
result: $\zeta(s)$ is meromorphic in □ with the only pole at $s_0 = 1$

Example 2

$$f(x) = \frac{1}{1+x} \underset{x \to 0^+}{=} \sum_{n=0}^{+\infty} (-1)^n x^n$$

$$\frac{1/x}{1+1/x} \underset{x \to +\infty}{=} \sum_{n=1}^{+\infty} (-1)^{n-1} x^{-n}$$

$$f^*(s) = \frac{\pi}{\sin \pi x}, \quad s \in \langle 0,1 \rangle$$



$$f^*(s) \asymp \sum_{n=0}^{+\infty} \frac{(-1)^n}{s+n}, \quad s \in \langle -\infty, 1 \rangle \text{ (continuation to the left of the f.s.)}$$

$$f^*(s) \asymp -\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{s-n}, \quad s \in \langle 0, +\infty \rangle \text{ (continuation to the right of the f.s.)}$$

$$f^*(s) \equiv \frac{\pi}{\sin \pi x} \asymp \sum_{n \in \Box} \frac{(-1)^n}{s+n} \quad (s \in \Box)$$

Example 3 (nonexplicit transform)

$$f(x) = \frac{1}{\sqrt{\cosh x}} = 1 - \frac{1}{4}x^2 + \frac{7}{96}x^4 - \frac{139}{5760}x^6 + \frac{5473}{645120}x^8 + \mathrm{O}(x^{10}) \quad x \to 0$$

$f^*(s)$ has a fundamental strip $\langle 0, +\infty \rangle$

$$f^*(s) \asymp \frac{1}{s} - \frac{1}{4}\frac{1}{s+2} + \frac{7}{96}\frac{1}{s+4} - \frac{139}{5760}\frac{1}{s+6} + \frac{5473}{645120}\frac{1}{s+8} \quad s \in \langle -10, +\infty \rangle$$

$$f^*(s) \asymp \frac{1}{s} - \frac{1}{4}\frac{1}{s+2} + \frac{7}{96}\frac{1}{s+4} - \frac{139}{5760}\frac{1}{s+6} + \frac{5473}{645120}\frac{1}{s+8} + \cdots \quad s \in \mathbb{C}$$

Converse mapping (1/3)

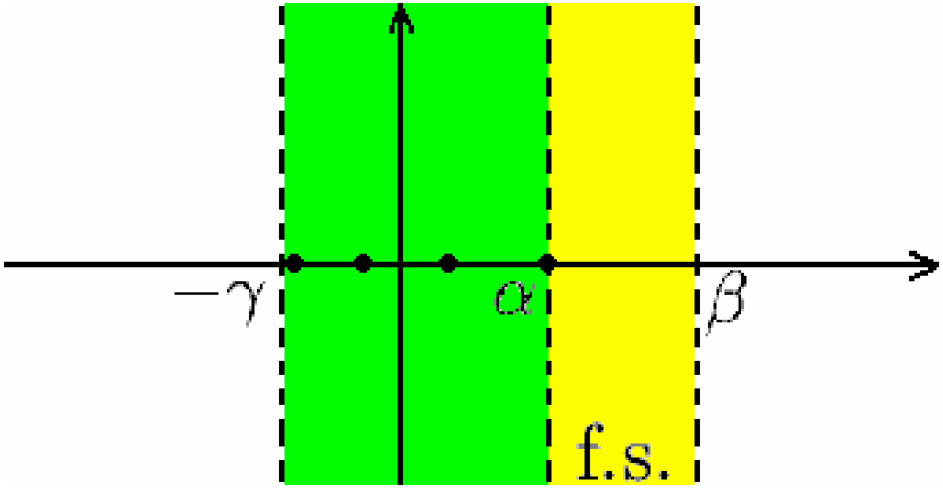
Theorem 3

Let $f(x) \in C(0, +\infty)$ have a transform $f^*(s)$ with nonempty fundamental strip $\langle \alpha, \beta \rangle$.

Let $f^*(s)$ be meromorphically continuable to $\langle -\gamma, \beta \rangle$ with a finite number of poles there, and be analytic on $\Re(s) = -\gamma$.

Let $f^*(s) = O\left(|s|^{-r}\right)$ with $r > 1$ when $|s| \to +\infty$ in $\langle -\gamma, \beta \rangle$.

…



Converse mapping (2/3)

Theorem 3 (continue)

…

If $$f^{*}(x) \asymp \sum_{(\xi,k)\in A} d_{\xi,k} \frac{1}{(s-\xi)^{k}} \quad s \in \langle -\gamma, \beta \rangle$$

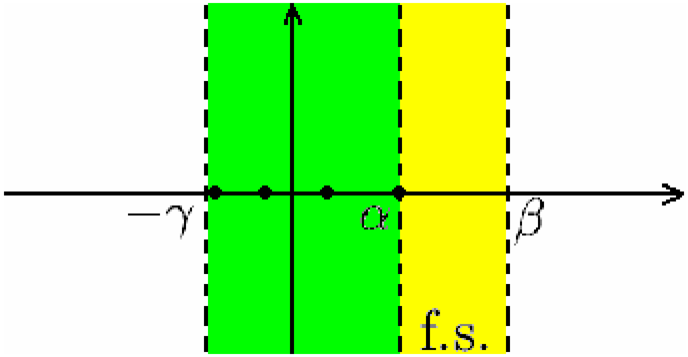
$$k \in \mathbb{N}, \quad -\gamma < -\xi \leq \beta$$

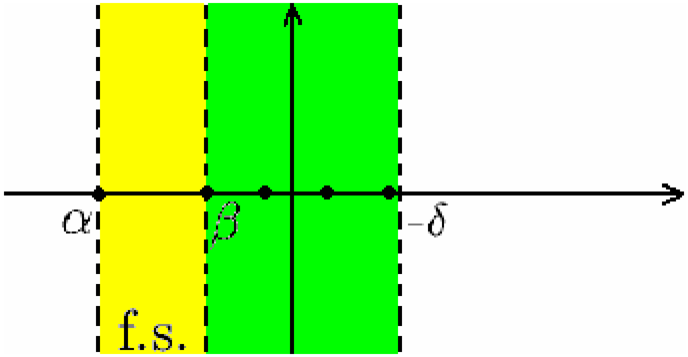
Then an asymptotic expansion of $f(x)$ at 0 is

$$f(x) = \sum_{(\xi,k)\in A} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^{k-1} \right) + \mathrm{O}(x^{\gamma})$$

Converse mapping (3/3)

$f^*(s)$	$f(x)$
Pole at ξ	Term in asymptotic expansion $\approx x^{-\xi}$
left of f.s.	expansion at 0
right of f.s	expansion at $+\infty$
Simple pole	
left: $\dfrac{1}{s-\xi}$	$x^{-\xi}$ at 0
right: $\dfrac{1}{s-\xi}$	$-x^{-\xi}$ at $+\infty$
Multiple pole	Logarifmic factor
left: $\dfrac{1}{(s-\xi)^{k+1}}$	$\dfrac{(-1)^k}{k!}x^{-\xi}(\log x)^k$ at 0
right: $\dfrac{1}{(s-\xi)^{k+1}}$	$-\dfrac{(-1)^k}{k!}x^{-\xi}(\log x)^k$ at $+\infty$





Example 4

$$\phi(s) = \frac{\Gamma(s)\Gamma(\nu - s)}{\Gamma(\nu)}, \text{ analytic in strip } \langle 0, \nu \rangle$$

$$\phi(s) \asymp \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \frac{\Gamma(\nu + j - 1)}{\Gamma(\nu)} \frac{1}{s + j} \quad s \in \langle -\infty, \nu \rangle$$

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{\nu/2 - \mathrm{i}\infty}^{\nu/2 + \mathrm{i}\infty} \phi(s) x^{-s} ds \text{ - original function}$$

$$f(x) = \sum_{j=0}^{M} \frac{(-1)^j}{j!} \frac{\Gamma(\nu + j - 1)}{\Gamma(\nu)} x^j + \mathrm{O}(x^{M+1/2}) \quad x \to 0$$

$$\overline{f}(x) = (1 + x)^{-\nu} \text{ has the same expansion at 0}$$

$$f(x) = (1 + x)^{-\nu} + \varpi(x), \text{ where } \varpi(x) = \mathrm{O}(x^M) \quad \forall M > 0$$

Example 5

$$\phi(s) = \Gamma(1-s)\frac{\pi}{\sin \pi s} \quad s \in \langle 0,1 \rangle$$

$$\phi(s) \asymp \sum_{n=0}^{+\infty} (-1)^n n! \frac{1}{s+n} \quad s \in \langle -\infty, 1 \rangle$$

$$f(x) \sim \sum_{n=0}^{+\infty} (-1)^n n! x^n$$ - the expansion is only asymptotic!

$$\left(\quad f(x) = \sum_{n=0}^{M} (-1)^n n! x^n + O(x^{M+1/2}) \quad x \to 0 \right)$$

$$\left(\quad \text{In fact } f(x) = \int_0^{+\infty} \frac{e^{-t}}{1+xt} dt \quad \right)$$

Harmonic sums

Definition 3

$G(x) = \sum_j \lambda_j g(\mu_k x)$ harmonic sum

λ_j amplitudes

μ_j frequencies

$\Lambda(s) = \sum_j \lambda_j \mu_j^{-s}$ The Dirichlet series

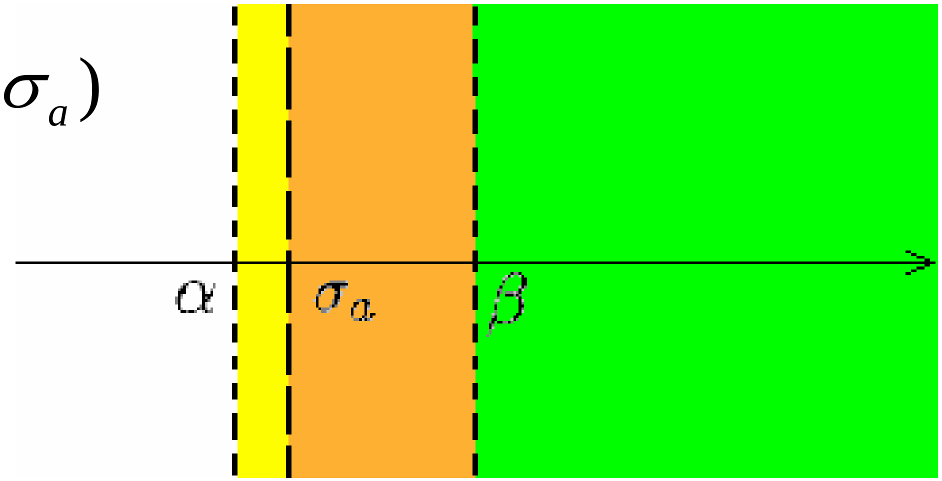
$M\left[g(\mu x); s\right] = \mu^{-s} g^*(s)$ (separation property)

$$\boxed{G^*(s) = \Lambda(s) g^*(s)}$$

Harmonic sum formula

The Mellin transform of the harmonic sum $G(x)=\sum_k \lambda_k g(\mu_k x)$ is defined in the intersection of the the fundamental strip of g(x) and the domain of absolute convergence of the Dirichlet series $\Lambda(s)=\sum_{\mathrm{k}} \lambda_k \mu_k{}^{-s}$ (which is of the form $\Re(s)>\sigma_a$ for some real σ_a)

$$G^*(s)=\Lambda(s)g^*(s)$$

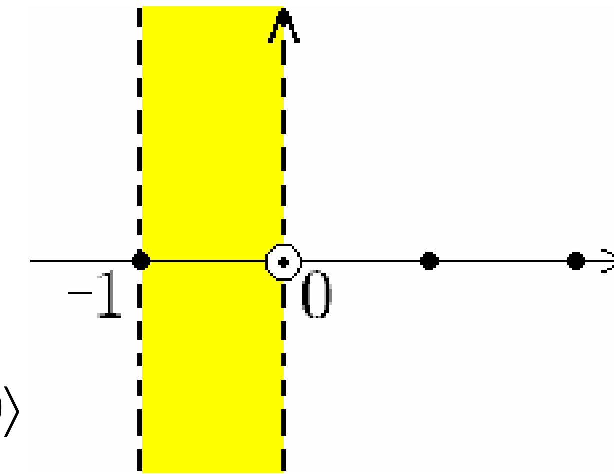


Example 6

$$h(x)=\sum_{k=1}^{+\infty}\left[\frac{1}{k}-\frac{1}{k+x}\right]=\sum_{k=1}^{+\infty}\frac{1}{k}\frac{x/k}{1+x/k},\quad \lambda_k=\frac{1}{k},\quad \mu_k=\frac{1}{k},\quad g(x)=\frac{x}{1+x}$$

$$h(n)=H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$\Lambda(s)=\sum_{k=1}^{+\infty}\lambda_k\mu_k^{-s}=\sum_{k=1}^{+\infty}k^{-1+s}=\zeta(1-s)$$



$$h^*(s)=\Lambda(s)g^*(s)=-\zeta(1-s)\frac{\pi}{\sin\pi s}\quad s\in\langle -1,0\rangle$$

$$\zeta(s)=\frac{1}{s-1}+\gamma+\cdots,\quad \zeta(1-s)=-\frac{1}{s}+\gamma$$

$$h^*(s)\asymp\left[\frac{1}{s^2}-\frac{\gamma}{s}\right]-\sum_{k=1}^{+\infty}(-1)^k\frac{\zeta(1-k)}{s-k}\quad s\in\langle -1,+\infty\rangle$$

$$\boxed{H_n\sim\log n+\gamma+\sum_{k\geq 1}\frac{(-1)^k B_k}{k}\frac{1}{n^k}\sim\log n+\gamma+\frac{1}{2n}-\frac{1}{12n}+\frac{1}{120n^4}+\cdots}$$

Example 7

$$l(x) = \log\Gamma(x+1) - \gamma x = \sum_{n=1}^{+\infty}\left[\frac{x}{n} - \log\left(1+\frac{x}{n}\right)\right] \quad s \in < -2, +\infty >$$

$$\lambda_n = 1, \quad \mu_n = \frac{1}{n}, \quad g(x) = x - \log(1+x)$$

$$\Lambda(s) = \sum_{n=1}^{+\infty} \lambda_k {\mu_k}^{-s} = \sum_{n=1}^{+\infty} n^s = \zeta(-s)$$

$$l^*(s) = \Lambda(s) g^*(s) = -\zeta(-s)\frac{\pi}{s\sin\pi s}, \quad s \in \langle -2, -1\rangle$$

Simple poles in positive integers
Double poles in 0 and -1

$$l^*(s) \asymp \left[\frac{1}{(s+1)^2} + \frac{1-\gamma}{(s+1)}\right] + \left[\frac{1}{2s^2} - \frac{\log\sqrt{2\pi}}{s}\right] + \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}\zeta(-n)}{n(s+n)}$$

$$l(x) = \log(x!) - \gamma x = x\log x - (1-\gamma)x - \frac{1}{2}\log x + \log\sqrt{2\pi} + \sum_{n=1}^{+\infty}\frac{B_{2n}}{2n(2n-1)}\frac{1}{x^{2n-1}}$$

$$\boxed{\log(x!) = \log\left(x^x e^{-x}\sqrt{2\pi x}\right) + \sum_{n=1}^{+\infty}\frac{B_{2n}}{2n(2n-1)}\frac{1}{x^{2n-1}}}$$

Example 8

$$L_w(x) = \sum_{n=1}^{+\infty} \frac{e^{-nx}}{n^w} \quad (\text{polylogarithm } Li_w(z) = \sum_{n=1}^{+\infty} z^n n^{-w}) \quad w = k \in \mathbb{N}$$

$$\lambda_n = \frac{1}{n^w}, \quad \mu_k = n, \quad g(x) = e^{-x}$$

$$\Lambda(s) = \zeta(s+k), \quad L_k^*(s) = \zeta(s+k)\Gamma(s) \quad s \in \langle 1, +\infty \rangle$$

$$L_k^*(s) \asymp \sum_{\substack{n=0 \\ n \neq k-1}}^{+\infty} (-1)^n \frac{\zeta(k-n)}{n!} \frac{1}{s+n} + \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{1}{(s+k-1)^2} + \frac{H_{k-1}}{s+k-1} \right]$$

$$s \in \langle -\infty, +\infty \rangle$$

$$L_k(x) = \frac{(-1)^{k-1}}{(k-1)!} x^{k-1} [-\log x + H_{k-1}] + \sum_{\substack{n=0 \\ n \neq k-1}}^{+\infty} (-1)^n \frac{\zeta(k-n)}{n!} x^n$$

$$L_{1/2}(x) = \sqrt{\frac{\pi}{x}} + \sum_{n=0}^{+\infty} (-1)^n \frac{\zeta(\frac{1}{2} - n)}{n!} x^n$$

Example 9

modified theta function

$$\Theta_w(x)=\sum_{n=1}^{+\infty}\frac{e^{-n^2x^2}}{n^w},\quad \lambda_n=\frac{1}{n^w},\quad \mu_n=n,\quad g(x)=e^{-x^2}$$

$$\Theta_1^{\,*}(s)=\frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s+1)\quad s\in\langle 0,+\infty\rangle$$

double pole at $s=0$ and simple poles at $s=-2m$

$$\Theta_1^{\,*}(s)\asymp\left[\frac{1}{s^2}+\frac{\gamma}{2s}\right]+\frac{1}{12}\frac{1}{s+2}+\frac{1}{240}\frac{1}{s+4}+\cdots\quad s\in\mathbb{C}$$

$$\Theta_1(x)=-\log x+\frac{\gamma}{2}+\frac{1}{12}x^2+\frac{1}{240}x^4+\cdots\quad x\to 0$$

$$\Theta_0^{\,*}(s)=\frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\quad \Theta_0=\frac{1}{2}\frac{\sqrt{\pi}}{x}-\frac{1}{2}+\mathrm{O}(x^M)\quad x\to 0$$

Example 10

$$D(x) = \sum_{k=1}^{+\infty} d(k) e^{-kx}, \quad \lambda_k = d(k), \quad \mu_k = k, \quad g(x) = e^{-x}$$

$$\Lambda(s) = \sum_{k=1}^{+\infty} \lambda_k \mu_k^{-s} = \sum_{k=1}^{+\infty} \frac{d(k)}{k^{-s}} = \zeta^2(s)$$

$$D^*(s) = \Gamma(s)\zeta^2(s)$$

$$D^*(s) \asymp \left[\frac{1}{(s-1)^2} + \frac{\gamma}{s-1}\right] + \left[\frac{1}{4s}\right] - \sum_{k=0}^{+\infty} \frac{\zeta^2(-2k-1)}{(2k+1)!} \frac{1}{s+2k+1}$$

$$D(x) \sim \frac{1}{x}(-\log x + \gamma) + \frac{1}{4} - \sum_{k=0}^{+\infty} \frac{\zeta^2(-2k-1)}{(2k+1)!} x^{2k+1}$$

The end